ON THE STABILITY OF HAMILTONIAN SYSTEMS IN THE PRESENCE OF RESONANCES

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Necessary and sufficient conditions are obtained for the stability of the equilibrium of Hamiltonian systems in the presence of resonances.

1. Formulation of the problem. The equilibrium of the Hamiltonian system

$$\frac{dx_\alpha}{dt} = \frac{\partial}{\partial y_\alpha} H(x, y), \quad \frac{dy_\alpha}{dt} = -\frac{\partial}{\partial x_\alpha} H(x, y) \quad (\alpha = 1, 2, \ldots, n) \tag{1.1}$$

is investigated.

Without limiting the generality, we shall consider the origin $x = (x_1, \ldots, x_n) = 0$, $y = (y_1, \ldots, y_n) = 0$ to be the equilibrium position. In this case the Hamiltonian function can be represented as follows:

$$H(x, y) = H_2(x, y) + H_3(x, y) + \ldots \tag{1.2}$$

Here $H_k(x, y)$ is a homogeneous polynomial of degree k.

The question of stability is examined for the case when it is not solved in a linear approximation. In other words it is assumed that:

a) The quadratic form $H_2(x, y)$ in (1.2) is indefinite (otherwise stability would follow from the Lagrange-Dirichlet theorem);

b) The eigenvalues of the linearized system are pure imaginaries (otherwise instability is assured by the Liapunov theorem).

Moreover, we shall assume that

c) There are no multiple eigenvalues among those $\beta_1, \beta_2, \ldots, \beta_n, -\beta_1, -\beta_2, \ldots, -\beta_n$ of the linearized system.

Under the assumptions listed, it can be considered that the quadratic form $H_2(x, y)$ in (1.2) is written as

$$H_2(x, y) = \frac{1}{2} \sum^n \beta_\alpha (x_\alpha^2 + y_\alpha^2) \tag{1.3}$$

Some integral relationships between the frequencies of the linearized system, the resonance relationships, play an important part in questions of stability.

Definition. It is said that the system (1.2) possesses resonance if there exists the integer vector

$$k = (k_1, k_2, \ldots, k_n) \neq 0, \qquad k_\alpha \geqslant 0$$

such that

$$k_1\beta_1 + k_2\beta_2 + \ldots + k_n\beta_n = 0$$

The number $|k| = k_1 + k_2 + \ldots + k_n$ is called the order of resonance. As is known [1, 2], in the absence of resonance in the system it is stable in any finite order; instability can result only in the presence of resonance.

The paper is devoted to an investigation of the stability of the equilibrium of Hamiltonian systems which are neutral (i.e. Conditions (a), (b) and (c) are satisfied) in a linear approximation, in the presence of resonances. All the investigations are carried out herein in two stages.

1. The investigation of truncated systems which results in the theorem: Necessary and sufficient for the Liapunov stability of a truncated system is the presence of an "invariant ray" among the solutions of this system, which would be the analog of the eigenfunction in the nonlinear situation.

2. Carrying the results on the stability of the truncated (model) system over for the original system (1.1).

Theorem 1.1. If there is an invariant ray among the solutions of the model system, then the system is Liapunov unstable, otherwise there is Birkhoff stability in any finite order.

Results of the analysis presented here confirm A. M. Molchanov's hypothesis formulated in his doctoral thesis, although it referred to a nonresonant situation and to non-Hamiltonian systems.

Molchanov hypothesis. Necessary and sufficient for the instability of a system is the presence of an invariant ray in the model system.

2. Third and fourth order resonances. 1°. Let the system (1.1) possess a third order resonance. The Hamiltonian (1.2) of the system (1.1) can be reduced to normal third order form by polynomial canonical transformations. Discarding terms above third order, the Hamiltonian of the truncated system is written as follows in polar canonical variables:

$$\Gamma = \sum_{\alpha=1}^{n} \beta_\alpha \rho_\alpha + 2A \sqrt{\rho^k} \cos \psi \tag{2.1}$$

$$k_1\beta_1 + k_2\beta_2 + \ldots + k_n\beta_n = 0, \quad \rho^k = \rho_1^{k_1}\rho_2^{k_2} \ldots \rho_n^{k_n}, \quad |k| = 3$$

$$\psi = k_1\varphi_1 + k_2\varphi_2 + \ldots + k_n\varphi_n, \qquad \alpha = 1, 2, \ldots, n$$

Here ρ_α and φ_α are canonical polar coordinates, k is an integral vector, ψ is the resonance phase. The system of equations corresponding to (2.1) is

$$\frac{d\rho_\alpha}{dt} = -2Ak_\alpha \sqrt{\rho^k} \sin \psi \qquad (\alpha = 1, \ldots, n) \tag{2.2}$$

$$\frac{d\psi}{dt} = -A\sqrt{\rho^k} \sum_{\alpha=1}^{n} \frac{k_\alpha^2}{\rho_\alpha} \cos \psi$$

We call the system (2.2) the model third order system. If $A \neq 0$, it is then said the resonance is included. It can be verified that the system (2.2) possesses the growing solution

$$\rho_\alpha(t) = k_\alpha b(t) \qquad (\alpha = 1, \ldots, n)$$

$$\psi(t) = \text{const} = -\frac{\pi}{2}\frac{|A|}{A}, \quad \frac{db}{dt} = 2|A|\sqrt{k^k}\, b^{3/2} \tag{2.3}$$

We call the solution (2.3) the invariant ray of the model system (2.2). The presence of the ray denotes the instability of (2.2). Thus, the theorem is proved.

Theorem 2.1. If the system (1.1) possesses one included resonance, it is then Birkhoff unstable in the third order.

2°. Now let the system (1.1) possess one resonance, and let its order be four. The Hamiltonian (1.2) can be reduced to normal form by a canonical, polynomial transformation. Discarding terms above fourth order, we obtain the model fourth order system

$$\Gamma = \sum_{\alpha=1}^{n} \beta_\alpha \rho_\alpha + 2A \sqrt{\rho^k} \cos\psi + A^{\alpha\beta}\rho_\alpha\rho_\beta \tag{2.4}$$

$$\frac{d\rho_\alpha}{dt} = -2Ak_\alpha \sqrt{\rho^k} \sin\psi \qquad (\alpha = 1, \ldots, n) \tag{2.5}$$

$$\frac{d\psi}{dt} = -A\sqrt{\rho^k} \sum_{\alpha=1}^{n} \frac{k_\alpha^2}{\rho_\alpha} \cos\psi - 2A^{\alpha\beta}k_\alpha\rho_\beta$$

$$(k_1\beta_1 + \ldots + k_n\beta_n = 0, \quad k_1 + \ldots + k_n = 4)$$

Theorem 2.2. Satisfaction of the condition

$$|A| < \frac{|A^{\alpha\beta}k_\alpha k_\beta|}{2\sqrt{k^k}} = S \tag{2.6}$$

is a sufficient condition for stability of (2.5).

Proof. The system (2.5) possesses the following integrals:

$$I_\alpha = \rho_\alpha - \frac{k_\alpha}{k_1}\rho_1 \quad (\alpha = 2, 3, \ldots, n), \quad F = \Gamma - \sum_{\alpha=1}^{n} \beta_\alpha\rho_\alpha \tag{2.7}$$

From these integrals let us construct a non-negative integral of the system

$$L = \sum_{\alpha=2}^{n} I_\alpha^4 + F^2 \tag{2.8}$$

which is a Liapunov function if it is positive definite, i.e. $L = 0$, only for $\rho = 0$. On the invariant surface $I_2^4 + \ldots + I_n^4 = 0$, which is described by the equations $\rho_\alpha = k_\alpha b(t)$ the integral F becomes

$$F = [2A\sqrt{k^k}\cos\psi + A^{\alpha\beta}k_\alpha k_\beta]\, b^2$$

Upon compliance with condition (2.6), the integral F vanishes only for $\rho = 0$. Therefore L is positive definite. The theorem is proved.

Remarkably, it turns out that condition (2.6) is also a necessary condition for stability.

Theorem 2.3. The equilibrium of the system (2.5) is unstable if the condition

$$|A| > S \tag{2.9}$$

is satisfied.

Proof. Let us construct a growing solution of the system (2.5) which we shall seek in the form of an invariant ray

$$\rho_\alpha(t) = k_\alpha b(t) \qquad (\alpha = 1, \ldots, n), \qquad \psi(t) = \text{const.} \tag{2.10}$$

If such a solution exists, then thereon

$$F = [2A\sqrt{k^k}\cos\psi + A^{\alpha\beta}k_\alpha k_\beta]\, b^2$$

Upon compliance with condition (2.9) a ψ_0 can be selected such that

$$\cos\psi_0 = -\frac{A^{\alpha\beta}k_\alpha k_\beta}{2A\sqrt{k^k}}, \qquad A\sin\psi_0 < 0$$

Hence $d\psi/dt = 0$ and $\psi = \psi_0 = \text{const}$. The equation for $b(t)$ becomes

$$db/dt = [(2A\sqrt{k^k})^2 - (A^{\alpha\beta}k_\alpha k_\beta)^2]^{1/2} b^2 \tag{2.11}$$

The instability of (2.5) under the condition (2.8) (*) follows from the presence of the growing solution (2.9), (2.10). The boundary case of stability $|A| = S$ is examined in the supplement.

3. Invariant ray and Liapunov instability.

Theorem 3.1. If the system (1.1) possesses one resonance and among the solutions of its model system ($|k| = 3$ or $|k| = 4$) there is an invariant ray, then the equilibrium of (1.1) is Liapunov unstable.

This theorem will be proved by using the known Chetaev algorithm. A function $P(\rho, \varphi)$ will be constructed such that the time derivative in its domain of nonpositivity $(P(\rho, \varphi) \leqslant 0)$ taken by virtue of the original system, is negative. Liapunov instability for (1.1) follows from the presence of the Chetaev function.

Proof. By a canonical polynomial transformation we reduce (1.2) to

$$\Gamma = \sum_{\alpha=1}^{n} \beta_\alpha \rho_\alpha + \Gamma_p(\rho, \psi) + A_l \rho^l + R(\rho, \psi) \tag{3.1}$$

Here $l = (l_1, \ldots, l_n)$ is the integral vector $l_\alpha \geqslant 0$, $|l| = l_1 + \ldots + l_n$, $2 < |l| < |k|$, the degree of $R(\rho, \varphi)$ in the variable ρ is greater than $|k|$.

$$\Gamma_p(\rho, \psi) = \begin{cases} 2A\sqrt{\rho^k}\cos\psi & (|k| = 3) \\ 2A\sqrt{\rho^k}\cos\psi + A^{\alpha\beta}\rho_\alpha\rho_\beta & (|k| = 4) \end{cases} \tag{3.2}$$

A model system of order $2|k|$ is described by the Hamiltonian

$$\Gamma_1 = \Gamma - R(\rho, \varphi) \tag{3.3}$$

The quantities

$$I_\alpha = \rho_\alpha - \frac{k_\alpha}{k_1}\rho_1 \quad (\alpha = 2, \ldots, n) \quad F = \Gamma_1 - \sum_{\alpha=1}^{n} \beta_\alpha \rho_\alpha$$

are integrals of the model system, i.e.

$$\{\Gamma_1, I_\alpha\} = \{\Gamma_1, F\} = 0, \qquad \{\ \} - \text{Poisson brackets}$$

Hence, there follows

$$\left|\frac{dI_\alpha}{dt}\right| = |\{I_\alpha, \Gamma\}| = |\{I_\alpha, R\}| < |\rho|^{|k|+1/2} \tag{3.4}$$

Here $|\rho| = \rho_1 + \cdots + \rho_n$, and the symbol $f(\rho) < \rho^m$ means that the expansion of $f(\rho)$ in terms of ρ contains only powers of order not less than m.

Now, let us examine the function

$$P(\rho, \psi) = \sum_{\alpha=2}^{n} I_\alpha^2 + F^2 - \varkappa^2 \rho_1^{|k|} \tag{3.5}$$

Let us introduce the following notation for the domains:

$$\Omega(r) = \{P(\rho, \psi) \leqslant 0,\ \rho_1 < r\} \tag{3.6}$$

$$\Omega_1 = \left\{\sum_{\alpha=2}^{n} I_\alpha^2 \leqslant \varkappa^2 \rho_1^{|k|},\ \rho_1 < r\right\} \tag{3.7}$$

$$\Omega_\psi = \{F^2 \leqslant \varkappa^2 \rho_1^{|k|},\ \rho_1 < r\} \tag{3.8}$$

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*) An analogous result has been obtained in [3] for systems with two degrees of freedom.

In this notation

$$\Omega(r) \subset \Omega_I \cap \Omega_\psi \tag{3.9}$$

It follows from (3.7) that

$$|I_\alpha| \leqslant \varkappa \rho_1^{1/2|k|}$$

or

$$\rho_1 \left[\frac{k_\alpha}{k_1} - \varkappa \rho_1^{1/2|k|-1}\right] \leqslant \rho_\alpha \leqslant \rho_1 \left[\frac{k_\alpha}{k_1} + \varkappa \rho_1^{1/2|k|-1}\right] \tag{3.10}$$

is satisfied everywhere in Ω_I. If we consider (3.9) in Ω_I for $\rho_1 \to 0$, then for $|k| \geqslant 3$ we obtain

$$\rho_\alpha = \frac{k_\alpha}{k_1} \rho_1 [1 + o(1)] \tag{3.11}$$

As $\rho_1 \to 0$ the inequality (3.4) in Ω_I goes over into

$$\left| \frac{dI_\alpha}{dt} \right| = o(\rho_1^{|k|}) \tag{3.12}$$

Let us now show that $\Omega(r)$ separates into the sum of two closed domains

$$\Omega_+(r) = \{P(\rho, \varphi) \leqslant 0, \ \rho_1 < r, \ \sin\psi > 0\}$$

$$\Omega_-(r) = \{P(\rho, \varphi) \leqslant 0, \ \rho_1 < r, \ \sin\psi > 0\}$$

for sufficiently small r, so that

$$\Omega(r) = \Omega_-(r) \cup \Omega_+(r), \qquad \Omega_+(r) \cap \Omega_-(r) = \{\rho = 0\}$$

To do this it is sufficient to show that $\sin\psi \neq 0$ in $\Omega(r)$ for sufficiently small r. Let us assume the opposite and arrive at a contradiction. Thus, let $\sin\psi = 0$ for arbitrarily small r.

We carry out the subsequent reasoning separately for $|k| = 3$ and $|k| = 4$.

The case $|k| = 3$, $A \neq 0$.

Under the assumption made, it follows from (3.8)

$$4A^2\rho^{|k|} \leqslant \varkappa^2\rho_1^3 \tag{3.13}$$

Let us consider (3.12) in Ω_I for $\rho_1 \to 0$, we obtain

$$4A^2 \frac{k^3}{k_1^3} \rho_1^3 [1 + o(1)] \leqslant \varkappa^2 \rho_1^3 \tag{3.14}$$

Selecting $\varkappa^2 = 2A^2k^k / k_1^3$, we see that (3.13) is not satisfied for small enough r.

The case $|k| = 4$, $2A\sqrt{k^k} - |A^{\alpha\beta}k_\alpha k_\beta| = \delta^2$

Under the assumption made, there results from (3.8)

$$|\pm 2A\sqrt{\rho^k} + A^{\alpha\beta}\rho_\alpha\rho_\beta| < \varkappa\rho_1^2 \tag{3.15}$$

Let us find the lower bound of the left side of (3.15)

$$|\pm 2A\sqrt{\rho^k} + A^{\alpha\beta}\rho_\alpha\rho_\beta| \geqslant \big|2|A|\sqrt{\rho^k} - |A^{\alpha\beta}\rho_\alpha\rho_\beta|\big| \tag{3.16}$$

Substituting into (3.15), we obtain

$$\big|2|A|\sqrt{\rho^k} - |A^{\alpha\beta}\rho_\alpha\rho_\beta|\big| < \varkappa\rho_1^2 \tag{3.17}$$

Considering (3.17) in Ω_I for $\rho_1 \to 0$ and taking account of (3.10) we see that

$$k_1^{-2}\big[\,\big|2|A|\sqrt{k^k} - |A^{\alpha\beta}k_\alpha k_\beta| + o(1)\big|\,\rho_1^2 = k_1^{-2}[\delta^2 + o(1)]\,\rho_1^2 < \varkappa\rho_1^2$$

is not satisfied for small enough r if we select

$$\varkappa = {}^1/_2 k_1^{-2}\delta^2$$

Now, let us examine the derivative

$$\frac{dP}{dt} = \{P, \Gamma\} = \{P, \Gamma_1\} + \{r, R\} = 2\sum_{\alpha=2}^{n} I_\alpha \{I_\alpha, R\} + \tag{3.18}$$

$$+ 2F\{F, \Gamma_1\} + 2F\{F, R\} - \varkappa^2 |k| \rho_1^{|k|-1}\{\rho_1, \Gamma\}$$

Let us estimate the order of magnitude of the quantities in the right side of (3.18) in Ω_I for $\rho_1 \to 0$

$$|I_\alpha \{I_\alpha, R\}| = o(\rho_1^{3/2|k|}), \qquad |F\{F, \Gamma_1\}| = o(\rho_1^{3/2|k|-1/2})$$

$$\{\rho_1, \Gamma\}\rho_1^{|k|-1} \sim \rho_1^{3/2|k|-1} \tag{3.19}$$

Taking account of (3.18), (3.19) in Ω_I for $\rho_1 \to 0$, we can rewrite (3.18) as follows:

$$dP / dt = 2\varkappa^2 k_1 |k| \rho_1^{3/2|k|-1} A \sin\psi [1 + o(1)] \tag{3.20}$$

It hence follows that $dP / dt \leqslant 0$ is one of the domains $\Omega_+(r)$ or $\Omega_-(r)$ where $A \sin\psi <$ < 0 for sufficiently small r. Hence $dP / dt = 0$ only when $\rho = 0$.

Thus, the function

$$P(\rho, \psi) = \sum_{\alpha=2}^{n} I_\alpha^2 + F^2 - \varkappa^2\rho_1^{|k|}$$

$$\varkappa^2 = \begin{cases} 2A^2 k^k / k_1^3 & (|k| = 3) \\ {}^1/_4\, \delta^2 / k_1^2 & (|k| = 4) \end{cases}$$

is the Chetaev function of the system under consideration. The theorem is proved.

Theorem 3.2. If the system (1.1) possesses one resonance and its order is four ($|A| < S$), then from the stability of the system in the fourth order ($|k| = 4$) follows the Birkhoff stability of (1.1) in any order.

The construction of Liapunov function of a model system of arbitrary order is completely analogous to the construction in Sect. 2.

Theorem 3.3. If besides resonance of order p ($p \leqslant 4$), the system (1.1) contains resonances of higher orders, then Theorem 3.1 remains valid.

The proof of this theorem is completely analogous to the proof of Theorem 3.1. In this case the Chetaev function is constructed as follows:

$$P(\rho, \psi) = \sum_{\alpha=2}^{n} I_\alpha^{2s} + F^{2m} - \rho_1^3$$

$$1 < s < {}^3/_2, \qquad {}^3/_4 < m < 1$$

for $|k| = 3$.

For fourth order resonance

$$P(\rho, \psi) = \sum_{\alpha=2}^{n} I_\alpha^{2s} + F^2 - \varkappa^2\rho_1^4$$

$${}^4/_3 < s < 2, \quad \varkappa = {}^1/_2\, \delta^2 / k_1^2$$

4. Higher order resonances ($|k| \geqslant 5$). If the order of the single resonance in the system is $p \geqslant 5$, then the Hamiltonian of the model system of this order is written as

$$\Gamma = \sum_{\alpha=1}^{n} \beta_\alpha \rho_\alpha + A_l \rho^l + \Gamma_p(\rho, \psi) \tag{4.1}$$

Here $l = (l_1, \ldots, l_n)$ is an integral vector and

$$l_\alpha \geqslant 0 \quad |l| = l_1 + \cdots + l_n, \quad \gamma = [p - {}^1/_2], \quad 2 \leqslant |l| \leqslant \gamma; \quad p = |k|$$

$$\Gamma_p(\rho, \psi) = \begin{cases} 2A\sqrt{\rho^k}\cos\psi & (|k| = 2m+1,\ m \geqslant 2) \\ 2A\sqrt{\rho^k}\cos\psi + A_l\rho^l & (|k| = 2m,\ |l| = m,\ m > 2) \end{cases} \tag{4.2}$$

$$k_1\beta_1 + \ldots + k_n\beta_n = 0$$

Here $k = (k_1, \ldots, k_n)$ is the resonance vector. It is easy to verify that the conditions

$$A_l k^l = 0, \qquad |l| = 2, 3, \ldots, \gamma \tag{4.3}$$

are necessary conditions for instability. Not satisfying at least one of the conditions (4.3) is a sufficient condition for the stability of (1.1) to any finite order (*). If conditions (4.3) are satisfied, then the question of stability is solved in complete analogy with lower order resonances. Thus, included odd order resonance always results in Liapunov instability upon compliance with the condition

$$|A| > \frac{|A_l k^l|}{2\sqrt{k^k}}, \qquad |k| = 2m,\ |l| = m$$

Otherwise, the stability is in any finite order.

The proofs of these assertions are completely analogous to the proofs of the preceding Sections.

The results of the preceding Sections can be formulated in the form of the following theorem.

F u n d a m e n t a l t h e o r e m. If the system (1.1) is neutral in a linear approximation and possesses one resonance ($|k| = p$), then the presence of an invariant ray among the solutions of the model system of order p is a necessary and sufficient condition for the Liapunov instability of (1.1). Otherwise (if there is no ray), there is Birkhoff stability in any finite order.

S u p p l e m e n t. B o u n d a r y c a s e o f s t a b i l i t y. Let us examine the boundary case of stability in an example of fourth order resonance since all the reasoning carries over to the case of arbitrary even order resonance.

Thus, the system possesses one fourth order resonance

$$\Gamma = \sum_{\alpha=2}^{n} \beta_\alpha \rho_\alpha + 2A\sqrt{\rho^k}\cos\psi + A^{\alpha\beta}\rho_\alpha\rho_\beta \tag{S.1}$$

and the Hamiltonian of the model fourth order system

$$|A\cdot = \frac{|A^{\alpha\beta}k_\alpha k_\beta|}{2\sqrt{k^k}} = S \tag{S.2}$$

Condition (S.2) corresponds to the boundary case of stability. Let us first note that in this case $\rho = 0$ is not an isolated stationary point. In fact, the invariant ray consists completely of fixed points in this case. It is hence natural to expect that the question of the stability of the equilibrium of (1.1) is not solved in the fourth order, but depends essentially on the higher members. Such is the situation. To illustrate this, let us present

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*) For $|l| = 2$ this condition follows from [4].

an example of a system unstable in the fourth order but stable in the eighth.

Let us consider a system with the Hamiltonian

$$\Gamma = \Gamma_1 + \Gamma_2$$

$$\Gamma_1 = \rho_1 - 3\rho_2 + 2\sqrt{3}\sqrt{\rho_1^3\rho_2}\cos\psi + 2\rho_1^2 \tag{S.3}$$

$$\Gamma_2 = 2\rho_1^2\rho_2\cos 2\psi + \rho_1^4$$

The Hamiltonian Γ_1 describes a model fourth order system. Let us select the initial data in this system such that the value of the integral $I = \rho_1 - 3\rho_2$ would be negative $(I < 0)$, and such that the integral of the system

$$F = 2\sqrt{3}\sqrt{\rho_1^3\rho_2}\cos\psi + 2\rho_1^2 |_{t=0} = 0$$

This can be done upon compliance with condition (S.2). We obtain the equation for ρ_1 as

$$d\rho_1 / dt = 6\sqrt{3}\,|I|^{1/2}\rho_1^{3/2}$$

Instability of the system (S.3) in the fourth order hence follows.

Let us show now that the equilibrium of (S.3) is stable. Let us construct the nonnegative integral

$$L = I^4 + G^2, \quad G = \Gamma - I$$

where G becomes on the invariant surface $\rho_1 = 3\rho_2$

$$G = [18\,(1 + \cos\psi) + (54\cos 2\psi + 81)\,\rho_2^2]\,\rho_2^2$$

Thus $L = 0$ only for $\rho_1 = \rho_2 = 0$, and is thereby a Liapunov function. The existence of systems unstable in higher orders than the fourth is evident in this case.

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